An Efficient Variable Screening Method for Effective Surrogate Models for Reliability-Based Design Optimization

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Abstract

In the reliability-based design optimization (RBDO) process, surrogate models are frequently used to reduce the number of simulations because analysis of a simulation model takes a great deal of computational time. On the other hand, to obtain accurate surrogate models, we have to limit the dimension of the RBDO problem and thus mitigate the curse of dimensionality. Therefore, it is desirable to develop an efficient and effective variable screening method for reduction of the dimension of the RBDO problem. In this paper, requirements of the variable screening method for deterministic design optimization (DDO) and RBDO are compared, and it is found that output variance is critical for identifying important variables in the RBDO process. An efficient approximation method based on the univariate dimension reduction method (DRM) is proposed to calculate output variance efficiently. For variable screening, the variables that induce larger output variances are selected as important variables. To determine important variables, hypothesis testing is used in this paper so that possible errors are contained in a user-specified error level. Also, an appropriate number of samples is proposed for calculating the output variance. Moreover, a quadratic interpolation method is studied in detail to calculate output variance efficiently. Using numerical examples, performance of the proposed method is verified. It is shown that the proposed method finds important variables efficiently and effectively.

Keywords: Variable Screening, RBDO, Surrogate Model, Output Variance, 1-D Surrogate Model, Partial Output Variance, Hypothesis Testing, Univariate Dimension Reduction Method

1. Introduction

The variable screening method is a useful method in the design optimization process because it can select essential design variables for accurate surrogate models and effective design optimization. In the formulation of a design optimization problem, a set of design variables that describe the system need to be identified [1]. Design variables are selected to be independent of each other as much as possible in the design space. The number of

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14. ABSTRACT

In the reliability-based design optimization (RBDO) process, surrogate models are frequently used to reduce the number of simulations because analysis of a simulation model takes a great deal of computational time. On the other hand, to obtain accurate surrogate models, we have to limit the dimension of the RBDO problem and thus mitigate the curse of dimensionality. Therefore, it is desirable to develop an efficient and effective variable screening method for reduction of the dimension of the RBDO problem. In this paper, requirements of the variable screening method for deterministic design optimization (DDO) and RBDO are compared, and it is found that output variance is critical for identifying important variables in the RBDO process. An efficient approximation method based on the univariate dimension reduction method (DRM) is proposed to calculate output variance efficiently. For variable screening, the variables that induce larger output variances are selected as important variables. To determine important variables, hypothesis testing is used in this paper so that possible errors are contained in a user-specified error level. Also, an appropriate number of samples is proposed for calculating the output variance. Moreover, a quadratic interpolation method is studied in detail to calculate output variance efficiently. Using numerical examples, performance of the proposed method is verified. It is shown that the proposed method finds important variables efficiently and effectively.

15. SUBJECT TERMS **Variable Screening, RBDO, Surrogate Model, Output Variance, 1-D Surrogate Model, Partial Output Variance, Hypothesis Testing, Univariate Dimension Reduction Method**

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independent design variables is known as the degrees of freedom, and this is the dimensionality of the optimization problem. To obtain an appropriate optimum design, a minimum number of design variables is required. For this reason, it is better to identify as many design variables as possible and then fix some of the variables at certain values according to the variable screening result. The variable screening method can play a key role, especially in reliability-based design optimization (RBDO), because the RBDO process requires a larger number of analyses than the deterministic design optimization (DDO) process due to reliability analyses and the design sensitivities of probabilities of failure. To this end, surrogate models are usually used to reduce the number of analyses required in RBDO. Various surrogate model methods such as the radial basis function (RBF), polynomial response surface (PRS), support vector regression (SVR), Kriging, and dynamic Kriging (DKG) methods have been developed [2-10]. However, even for the surrogate model, the number of design variables becomes a critical factor because surrogate model generation is difficult for high-dimensional problems, due to the curse of dimensionality.

Variable screening methods have been developed in various disciplines. In statistics, important variables were found to create an accurate surrogate model of computer simulation using the maximum likelihood estimator (MLE) of correlation parameters of the Gaussian process for a deterministic problem [11]. Using a regression model, essential variables among candidate variables were efficiently identified based on data [12, 13]. Especially in statistical learning theory, various feature selection methods have been developed to choose a reduced number of input variables to represent an output effectively [14]. In addition, methods such as manifold learning have been used to preserve input information in reduced dimension for efficient statistical analysis [15]. In physics, a variable screening model was developed for the quasi-molecular treatment of ion-atom collision [16]. In engineering, a confidence interval of the coefficient of a linear surrogate model was proposed to detect key variables for a car crash DDO problem [17]. A sampling-based sensitivity measure using a small amount of data was introduced to rank the importance of variables and was applied to long-term performance of a geologic repository for high-level radioactive waste [18]. Moreover, the design sensitivity method can be extended to the variable screening method because vital variables have larger design sensitivity. In the deterministic problem, the design sensitivity, which shows the rate of change in the performance measure at the design point, can be obtained using various methods [19, 20] and is called local sensitivity analysis (LSA) [21, 22]. For a reliability problem, the variability of the input random variable should be incorporated to assess the design sensitivity of a probabilistic constraint. The design sensitivity of the probabilistic constraint using the first-order reliability method (FORM) [23-25], dimension reduction method (DRM) [26, 27], and sampling-based stochastic sensitivity [28, 29], could be used to identify important design variables. In addition, global sensitivity analysis (GSA), such as correlation ratio [30], global sensitivity indices [31], and analytical GSA methods [22], can be used for variable screening as well [32].

However, previous works may have limitations to being directly applied to RBDO with surrogate models. If a method depends entirely on existing data [12-15], it may not be possible to carry out RBDO because design variables change during the optimization process. Finding input variables from among all the variables that may be irrelevant to an output to well represent the output from data [14] is not an issue from the RBDO perspective. The relationship and relative input variables are already known through the computer aided engineering (CAE) such as the finite element method (FEM) or computational fluid dynamics (CFD). A method using CAE to find variables

which significantly affect output reliability is more interesting. Moreover, capturing the input information in reduced variables [15] is not an issue with RBDO, either; how much output uncertainty is affected by the input variables is the main interest. A method developed for a specific problem [16] will be inadequate for broad applications. Variable screening and design sensitivity methods for a deterministic problem [11, 17, 19, 20] may not be applicable for RBDO because input randomness is not considered. Methods that require a very large number of analyses [30, 31] could be ineffective for RBDO of computationally demanding problems and could become unstable when sufficient numbers of analyses are not provided [18]. The design sensitivity of the probabilistic constraint using FORM or DRM [23-27] requires searching for the most probable point (MPP), which may be very difficult to obtain for a large-dimensional problem. If a method was developed based on the assumption that accurate full-dimensional surrogate models are available *a priori* [22, 28, 29], RBDO could be carried out using the surrogate models because it can provide accurate response and sensitivity of the problem, unless the optimization algorithm has a limitation on the number of design variables, which is not common. From the previous works, key desirable properties of a variable screening method for RBDO with a surrogate model were found: it should (1) be efficient, (2) consider input randomness, (3) not require a full-dimensional surrogate model, and (4) be applicable to broader problems.

Therefore, the objective of this paper is to develop a variable screening method that can satisfy the above desirable properties. The reliability analysis in RBDO captures the output variability induced by the input variability and the sensitivity of the performance function. The variable that induces larger output variability is important in the RBDO process. In this paper, a partial output variance, which is the output variance when one design variable has variability while others are fixed at their mean, is used to find important design variables [33]. The partial output variance is simple to calculate and requires a 1-D surrogate model for each design variable. The method introduced in this paper has strengths and weaknesses. Its main strength is its efficiency and practical applicability. Its weakness is accuracy; the interactions between the design variables are not fully captured. However, practical applicability is the focus in this paper because it is very important for large-scale problems. In the following sections, the proposed method will be explained in detail, and its strengths and weaknesses will be fully discussed. To demonstrate the effectiveness of the proposed method, analytical examples and a large-scale industrial problem are used.

2. Variable Screening

As explained in the introduction, screening out variables means finding important variables among all random variables. Here, the word "important" could have different meanings depending on the problem we are dealing with. In the following two sections, the difference between variable screening for DDO and RBDO will be explained. Based on the difference, the required properties of variable screening for RBDO will be introduced.

2.1 Variable Screening for DDO

A DDO problem can be formulated as

$$\text{minimize} \qquad \text{cost}(\mathbf{d})$$

$$\text{subject to} \quad \begin{aligned} G_j(\mathbf{d}) &\leq 0, & j &= 1,...,NC \\ \mathbf{d}^L \leq \mathbf{d} &\leq \mathbf{d}^U, & \mathbf{d} &\in \mathbf{R}^{NDV} \end{aligned} \tag{1}$$

where $\mathbf{d}$, G_j, NC, and NDV are the design variable vector, jth constraint function, number of constraints, and number of design variables, respectively.

As stated before, in the DDO problem, the input design variables do not have uncertainty, and thus the design sensitivity can be used as a barometer to determine the importance ranking of design variables with respect to the performance measure. The question is: "Where should the importance ranking of design variables be determined?" or "Where should the design sensitivity be calculated?"

The LSA calculates the design sensitivity at a given design point [21, 22]. Usually, LSA is used to provide the direction of design movement in the optimization process. For variable screening, LSA can provide the importance ranking of design variables at the current design point. However, the importance ranking at the given design point could be different from the ranking at other design points if the performance measure is a nonlinear function of design variables. On the other hand, GSA is used to calculate overall design sensitivity on the entire design domain. The GSA is like averaged design sensitivity in the design domain. As it is an average, the importance ranking using GSA could mislead at specific points or even regions. Hence, LSA and GSA have advantages and disadvantages for variable screening [21].

2.2 Variable Screening for RBDO

A general RBDO problem can be formulated as

$$\text{minimize} \quad \text{cost}(\mathbf{d})$$

$$\text{subject to} \quad \begin{aligned} P_{F_j} &= P[G_j(\mathbf{X}) > 0] \leq P_{F_j}^{Tar} & j &= 1,...,NC \\ \mathbf{d}^L \leq \mathbf{d} &\leq \mathbf{d}^U, \ \mathbf{d} \in \mathbf{R}^{NDV}, \ \text{and} \ \mathbf{X} \in \mathbf{R}^{NRV} \end{aligned} \tag{2}$$

where $\mathbf{d}$, G_j, $P_{F_j}^{Tar}$, NC, NDV, and NRV are the design variable vector, jth constraint function, jth target probability of failure, number of constraints, number of design variables, and number of random variables, respectively.

In the RBDO process, design variable vector $\mathbf{d}$ is the mean vector of the corresponding random variable $\mathbf{X}$. Though the design variable $\mathbf{d}$ is deterministic, the design sensitivity for RBDO should consider the randomness of $\mathbf{X}$ because the constraints are based on the probabilistic performance measure $P[G_j(\mathbf{X})>0]$ as shown in Eq. (2). Therefore, the design sensitivity of the performance measure alone cannot be used as a barometer. The design sensitivity of the probabilistic performance measure can be obtained by using several methods, such as FORM [23-25], DRM [26, 27], and sampling-based stochastic sensitivity [28, 29]. The design sensitivity of the probabilistic performance measure can be used for variable screening. The design sensitivities by those methods are LSA because they provide different sensitivities at different designs. The GSA method is also applicable for variable screening in RBDO problems as it is in DDO problems. Again, both LSA and GSA methods have advantages and disadvantages.

The random parameters will not increase the dimensionality of the optimization problem because they are not random design variables. However, the surrogate model that includes random parameters is still required because

they affect the output distribution. The main objective of this paper is to select important design variables so that accurate surrogate models can be generated and, at the same time, an appropriate optimum design (i.e., not suboptimum) can be obtained in the RBDO process. Hence, once variable screening is done, the screened-out random design variables need to be fixed, not to be a random parameter. However, fixing a random variable as a deterministic variable will reduce the total output variability.

Consider a simple example:

$$X_i \sim N(5,3^2), \quad i = 1, 2, \dots, 10$$

$$Y = \sum_{i=1}^{10} X_i \sim N\left(50, \left(3\sqrt{10}\right)^2\right) \tag{3}$$

If the probabilistic performance measure is $P[Y > 60]$, then the reliability analysis result is

$$P[Y > 60] = 1 - \Phi\left(\frac{60-50}{3\sqrt{10}}\right) = 0.1459 \tag{4}$$

However, if one dimension is reduced by screening out $X_{10} = \mu_{10} = 5$ while the other variables remain random, then the probabilistic performance measure changes to

$$\tilde{Y} = \sum_{i=1}^{9} X_i + 5 \sim N(50, 9^2) \tag{5}$$

As a consequence, the reliability analysis result yields

$$P\left[\tilde{Y} > 60\right] = 1 - \Phi\left(\frac{60-50}{9}\right) = 0.1333 \tag{6}$$

From Eqs. (4) and (6), 0.0126 (1.26%) of the output variability is decreased by screening out one variable. A more fundamental problem is that the lost amount 1.26% cannot be estimated without the full-dimensional reliability analysis result of Eq. (4). On the other hand, let's assume that X_{10} has a smaller variance of one. Then, the full-dimensional reliability analysis yields

$$P(Y > 60) = 1 - \Phi\left(\frac{60-50}{\sqrt{82}}\right) = 0.1347 \tag{7}$$

From Eqs. (6) and (7), the difference is 0.0014 (0.14%), which could be acceptable. Therefore, in this case, X_{10} could be fixed at the mean value. As shown in the example, the output variability decreases if any random variable is fixed at a deterministic value. However, there are some variables that affect the output variability a small amount. The variable screening method for effective surrogate models for RBDO is to find those variables that have small effects on the output variability. It is noted that the random parameters are considered as much as the random design variables in this paper. Even though the random parameters are not changing during the RBDO process, they will influence the output variability. Hence they should be considered in the variable screening process, so that reliability of the performance measure can be accurately approximated using reduced dimension.

3. Variable Screening with 1-D Surrogate Model

The probability of failure cannot be solely determined by the output variability. To obtain accurate probability of failure, the output distribution is needed, that is, all statistical information of the output is required. However,

even though an input distribution is known, it is very difficult to obtain complete output distribution since the performance measure could be implicit, a non-linear function, or even both. For example, for a given normal input distribution, the output distribution could be bimodal as well as asymmetric. Consequently, it is impractical to select a reduced number of input variables based solely on probability of failure. As discussed in previous sections, a screened-out variable will be fixed at its mean value. Then the change of output mean will be minimized. As a result, the output variability becomes the measure that can determine a probability of failure. Of course, other statistical moments or parameters, such as skewness and kurtosis, could affect probability of failure. However, either of these statistical moments cannot be a measure by itself. For example, a variable that induces larger (or smaller) output skewness may not be an important variable, but it could be an important variable when it induces larger (or smaller) output skewness and very similar output variability. We could consider a combination of the moments as a measure, but there are too many possible combinations to consider. Hence, under the assumption that the output mean is similar, the output variability is chosen as the measure to select vital variables for RBDO in this paper.

The output variability can be quantified by the output variance as shown in the previous section. The exact output variance of a nonlinear implicit performance measure is very difficult to obtain. Hence, an approximated output variance is used in this paper. In the following sections, the output variance is decomposed into partial output variances, which are the output variances when each input variable is random and the others are fixed at their mean values. Then, a method to find the design variables that have a large impact on output variance is developed using a hypothesis testing.

3.1 Approximated Output Variance

A univariate DRM is a well-known approximation method for statistical moments using multiple 1-D integrations [34]. Consider a performance measure Y and its realization y subject to random input vector $\mathbf{X} = \{X_1, \ldots, X_N\}^{\mathrm{T}}$:

$$Y(\mathbf{X}) = Y(X_1, \ldots, X_N), \quad y(\mathbf{x}) = y(x_1, \ldots, x_N) \tag{8}$$

Define a function Y_i, which is the performance measure when X_i is random and other variables are fixed at their mean values, as

$$Y_i = Y(\mu_1, \ldots, \mu_{i-1}, X_i, \mu_{i+1}, \ldots, \mu_N) \tag{9}$$

The realization of the performance measure at the input mean point $\boldsymbol{\mu_X}$ is defined as

$$y_0 = y(\boldsymbol{\mu_X}) \tag{10}$$

The lth statistical moment of Y, which is approximated using the univariate DRM, is defined as [34]

$$m_l \cong E\left[\left\{\sum_i Y_i - (N-1)y_0\right\}^l\right] \tag{11}$$

Then, the output variance σ_Y^2 can be approximated as

$$\sigma_Y^2 \cong m_2 - m_1^2 = \sum_i \sigma_{Y_i}^2 + 2\sum_{i>j} \rho_{Y_i Y_j} \sigma_{Y_i} \sigma_{Y_j} \tag{12}$$

where $\sigma_{Y_i}^2$ is the variance of Eq. (9), which is the partial output variance when only X_i is random, and $\rho_{Y_iY_j}$ is the correlation coefficient between Y_i and Y_j. As shown in Eq. (12), the partial output variances $\sigma_{Y_i}^2$ are the main variables for approximating the output variance σ_Y^2. When $\sigma_{Y_i}^2$ is larger than other partial output variances, it takes the largest portion in the output variance σ_Y^2. Therefore, if some X_i produces larger partial output variance than others, then X_i should be selected as an important variable. It is noted that calculation of $\sigma_{Y_i}^2$ requires only 1-D integration, and thus only 1-D surrogate models are required.

Statistical correlation between X_i and X_j yields the term of $\rho_{Y_iY_j}\sigma_{Y_i}\sigma_{Y_j}$ in Eq. (12) and affects the output variance. When X_i and X_j are strongly correlated, one could be replaced by the other. To calculate the term $\rho_{Y_iY_j}\sigma_{Y_i}\sigma_{Y_j}$, a two-dimensional surrogate model is required. If there are only a few correlation pairs, calculating the correlation term could be affordable. However, with a practical point of view, the partial output variance $\sigma_{Y_i}^2$ is the focus in this paper. As we are looking for important variables, not the value of the output variance of σ_Y^2, the partial output variance would be enough for variable screening. In Fig. 1, contours of independent, positively correlated ($\rho = 0.8$) and negatively correlated ($\rho = -0.8$) probability density functions are shown. Correlation determines how the random variables are distributed inside the box (dotted line), whereas the size of the box is determined by variances of X_1 and X_2. It can be seen that the primary effect on output variance is the box size, and then distribution inside the box follows. Consequently, to perform variable screening efficiently, the first thing we need to consider is the box size, not the distribution of random variables inside the box. Hence, the correlation term is not considered in this paper for efficiency and practicality. It is noted that the statistical correlation between X_i and X_j will be considered in reduced-dimensional RBDO if both variables are selected.
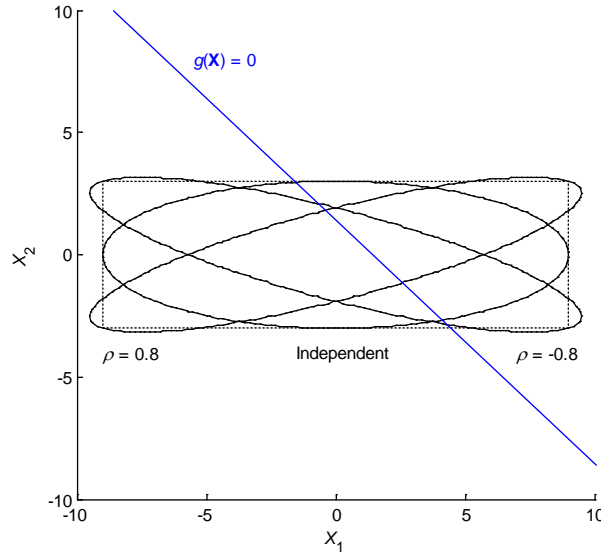

Figure 1. Effect of Variance and Correlation of Input Random Variables

The partial output variance of $\sigma_{Y_i}^2$ is like LSA because it can have different values at different input mean points $\boldsymbol{\mu_X}$, which is the current design point in the RBDO process. Hence, the variable screening result could be changed as the design point changes. There are several recommended points at which to perform variable screening using LSA. The first one is the DDO optimum. As the DDO optimum is usually close to the RBDO optimum, the variable screening result at the DDO optimum is likely to be similar to the result at the RBDO optimum. Also, the design point where most of the deterministic constraints are active can be a good candidate point. It is noted that DDO or the design point where the constraints are active could be obtained using the finite difference method in a practical engineering problem. Also, DDO could be achieved using the sensitivity obtained from a 1-D surrogate model because DDO requires only the deterministic LSA, which is 1-D.

3.2 Variable Screening Using Hypothesis Testing

Using the 1-D surrogate model, the partial output variance $\sigma_{Y_i}^2$ can be calculated approximately as

$$s_{Y_i}^2 = \frac{1}{ns-1} \sum_{j=1}^{ns} \left\{ y_i(\mu_1,\ldots,x_i^{(j)},\ldots,\mu_N) - \overline{y}_i \right\}^2 \tag{13}$$

where $x_i^{(j)}$ is the jth realization of the input random variable X_i, ns is the number of samples, and $\overline{y}_i$ is the mean of y_i as,

$$\overline{y}_i = \frac{1}{ns} \sum_{j=1}^{ns} y_i(\mu_1,\ldots,x_i^{(j)},\ldots,\mu_N) \tag{14}$$

As explained in previous sections, the partial output variance $s_{Y_i}^2$ can be used to determine important design variables. To make the variable screening procedure systematic, hypothesis testing is applied in this paper. Hypothesis testing can prevent undesirable choices that could occur during the decision-making procedure. Calculated partial output variance $s_{Y_i}^2$ depends on the number of samples ns. When ns is large enough, the variable screening result will be accurate. However, it would require a large computational time. Also, it is hard to determine what value of ns is large enough. When ns is small, it will include statistical error. If calculated $s_{Y_i}^2$ are distinctive from each other or with respect to the screening threshold value, then the effect of ns may not be significant. However, ns could cause an error when some $s_{Y_i}^2$ are similar to each other or are near the screening threshold value. Hypothesis testing can prevent this problem in a statistical manner by letting users control the error level.

Various hypothesis testing methods have been developed for the decision-making problem [35]. Among those methods, we need the one that is not sensitive to distribution type because the distribution type of Y_i or $s_{Y_i}^2$ is not known in general. The one-sample t-test is developed based on the central limit theorem, which states that the sample mean of non-normal distribution follows normal distribution approximately for a large number of samples.

The one-sample t-test is not sensitive to underlying distribution types, so it is used in this paper. As the t-test is a method for sample mean, $s_{Y_i}^2$ is calculated nr times for its statistical moments as

$$\bar{v}_i = \frac{1}{nr} \sum_{k=1}^{nr} s_{Y_i}^{2(k)} \tag{15}$$

$$s_{v_i}^2 = \frac{1}{nr-1} \sum_{k=1}^{nr} (s_{Y_i}^{2(k)} - \bar{v}_i)^2 \tag{16}$$

where $s_{Y_i}^{2(k)}$ is the kth repetition of $s_{Y_i}^2$ and nr is the number of repetitions. Now, the hypothesis is constructed:

$$H_0 : \bar{v}_i \le \mu_0 \quad \text{versus} \quad H_1 : \bar{v}_i > \mu_0 \tag{17}$$

where μ_0 is the criterion of hypothesis testing. According to Eq. (17), the design variable that corresponds to $\bar{v}_i$, which is greater than μ_0 (H_1 is true), will be selected as an important variable. Using the one-sample t-test, the hypothesis can be tested by checking the following statement:

$$\text{Reject } H_0 \text{ in favor of } H_1 \text{ if } q \ge t_{nr-1,1-\alpha} \tag{18}$$

where α is the significance level, $t_{nr-1,\bullet}$ is $t_{nr-1}^{-1}(\Box)$, and the test statistics q is defined as

$$q \equiv (\bar{v}_i - \mu_0) \Big/ \frac{s_{v_i}}{\sqrt{nr}} \tag{19}$$

In Eqs. (15) and (16), the uncertainty induced by ns is transferred to nr. Hence, ns can be a fixed number, whereas nr should be decided appropriately. Also, μ_0 needs to be identified in Eq. (17). μ_0 is the key criterion that decides important variables, and it should be a value relative to $\bar{v}_i$ because the relative difference of partial output variances should be checked for variable screening. At the same time, μ_0 needs to be statistically independent from $\bar{v}_i$ for reasonable hypothesis testing. In this paper, preliminary testing is proposed to obtain reasonable nr and μ_0 as follows. First, choose nr_0, which is large enough so that the central limit theorem holds. Then, calculate the initial statistical moments of $s_{Y_i}^2$ as

$$\bar{v}_i^{(0)} = \frac{1}{nr_0} \sum_{k=1}^{nr_0} s_{Y_i}^{2(k)} \tag{20}$$

$$s_{v_i}^{2(0)} = \frac{1}{nr_0-1} \sum_{k=1}^{nr_0} (s_{Y_i}^{2(k)} - \bar{v}_i^{(0)})^2 \tag{21}$$

Using the value from Eq. (20), the testing criterion μ_0 relative to $\bar{v}_i$ can be calculated as

$$\mu_0 = \frac{\gamma}{N} \sum_{i=1}^{N} \bar{v}_i^{(0)} \tag{22}$$

where γ is a constant that the user selects. nr is calculated by limiting type II error (H_0 is accepted when H_1 is true) at the level of false negative rate β as [35]

$$nr = \max \left(\frac{s_{v_i}^2 (t_{nr_0-1,1-\alpha} + t_{nr_0-1,1-\beta})}{\bar{v}_i^{(0)} - \mu_0}, nr_0 \right) \tag{23}$$

In Eq. (23), $t_{nr-1,\bullet}$ should be used instead of $t_{nr_0-1,\bullet}$ for accurate calculation of nr. However, Eq. (23) requires the value of nr on the right side to calculate nr. To avoid this problem, $t_{nr_0-1,\bullet}$ is used instead, and $t_{nr_0-1,\bullet}$ produces a conservative result as it is larger than $t_{nr-1,\bullet}$ because nr is larger than nr_0 in Eq. (23) and α is small. Finally, nr and μ_0 are determined so that the proposed hypothesis testing can be utilized.

3.3 1-D Surrogate Model

In previous sections, the 1-D surrogate model is treated as the given one because it is not difficult to generate. However, efficiently creating a 1-D surrogate model could be an issue. For efficiency, quadratic interpolation is proposed as a basic 1-D surrogate model in this paper. Quadratic interpolation may not be an adequate method for creating a surrogate model for a highly nonlinear performance measure. However, a nonlinear performance measure can be effectively approximated by a quadratic function on a small region. If X follows normal distribution, the domain of X is $(-\infty, \infty)$, whereas 99.73% of X is in $(\mu_X - 3\sigma_X, \mu_X + 3\sigma_X)$, which is much smaller than the infinite domain. Even if X does not follow normal distribution, the region $(\mu_X - 3\sigma_X, \mu_X + 3\sigma_X)$ can cover almost all (approximately 98%) of X. In view of the fact that this paper is focused on calculation of partial output variance, the region $(\mu_X - 3\sigma_X, \mu_X + 3\sigma_X)$ is large enough. Hence, the 1-D surrogate model needs to be accurate in the region $(\mu_X - 3\sigma_X, \mu_X + 3\sigma_X)$ so that quadratic interpolation could be an appropriate method to approximate the performance measure in the region.

Quadratic interpolation requires three samples, and the location of samples affects the accuracy of interpolation. The location of samples determined using the Chebyshev polynomial is known to give uniform error in the domain [36]. Because only the region $(\mu_X - 3\sigma_X, \mu_X + 3\sigma_X)$ is of interest, the location of samples is determined as $x_1 = \mu_X - 2.5981\sigma_X$, $x_2 = \mu_X$ and $x_3 = \mu_X + 2.5981\sigma_X$ using the Chebyshev polynomial. Since a random variable X may not be evenly distributed in its domain, providing uniform error does not necessarily mean that the calculated partial output variance is accurate. However, since no unique location of samples is best for accurate partial output variance, the sample location by the Chebyshev polynomial is used in this paper due to the fact that it yields reasonable results for various distribution types of the random variable X. If the random variable X has a closed and bounded domain like $[a, b]$, the domain can be directly used for calculation of partial output variance, and the location of samples are $x_1 = 0.93301a + 0.06699b$, $x_2 = (a + b)/2$ and $x_3 = 0.06699a + 0.93301b$, using Chebyshev polynomials.

To check the performance of a selected location of samples, a nonlinear performance measure Y is used as

$$Y(X) = 0.3 + \sin(16X/15 - 0.7) + \sin^2(16X/15 - 0.7) \tag{24}$$

Assuming that random variable X follows $N(0.5, 0.333^2)$, three locations of samples are chosen to compare the accuracy of the partial output variance. The first location is {0.167, 0.5, 0.833}, which is μ_X and $\mu_X \pm \sigma_X$, and the second location is from the Chebyshev polynomial as {−0.365, 0.5, 1.365}. The third location is wider, as {−0.667, 0.5, 1.667}, which is μ_X and $\mu_X \pm 3.5\sigma_X$. Partial output variances are calculated using 100,000 realizations of X, and

true partial output variance is calculated by Eq. (24) with the same realizations. To check the accuracy of the quadratic interpolation itself, mean square error (MSE) is calculated in the region of (−0.5, 1.5), which is ($\mu_X - 3\sigma_X$, $\mu_X + 3\sigma_X$) with 100 uniformly distributed points. The calculated result is shown in Table 1, and the shape of quadratic interpolations is shown in Fig. 2, where asterisk marks (*) represent the sample point. As shown in Table 1, the location of the sample using Chebyshev polynomials produces more accurate partial output variance compared to the true one and less MSE than the other cases.

Table 1. Quadratic Interpolation with Different Sample Locations

Location of samples		Quadratic interpolation			True
		{0.167, 0.5, 0.833}	{−0.365, 0.5, 1.365}	{−0.667, 0.5, 1.667}	
Partial output variance	Value	8.74E−02	7.84E−02	6.92E−02	7.84E−02
	Accuracy	111.5%	100.0%	88.3%	100.00%
MSE		1.17E−02	1.19E−03	9.00E−03	0
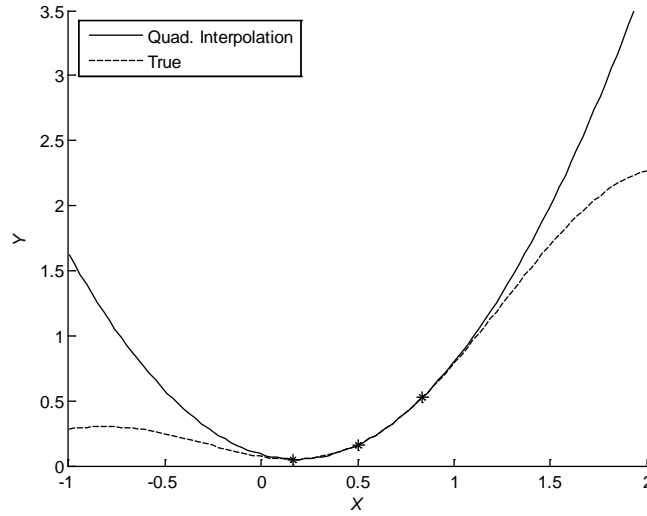
(a) Location of sample: {0.167, 0.5, 0.833}
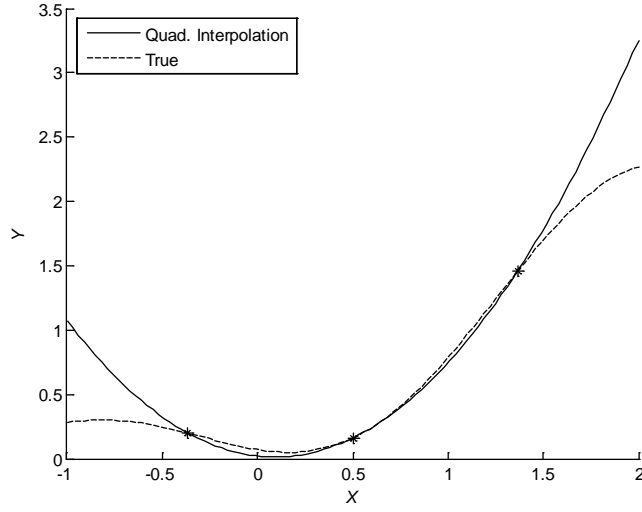
(b) Location of sample: {−0.365, 0.5, 1.365}
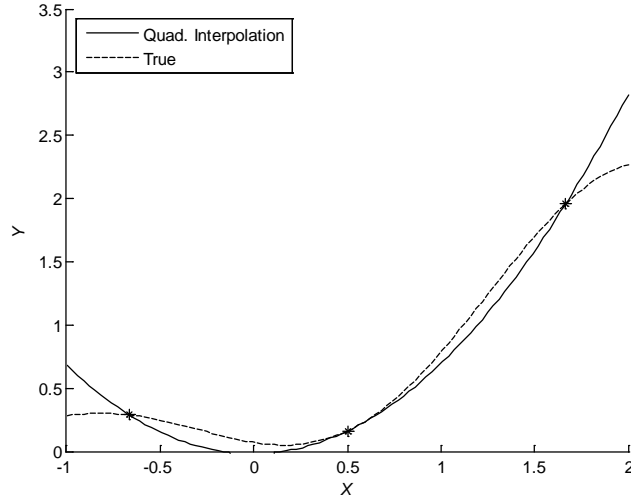
(c) Location of sample: {−0.667, 0.5, 1.667}

Figure 2. Quadratic Interpolation of *Y* with Different Locations of Samples

This example cannot represent all performance measures. When a highly nonlinear performance measure is expected, more sophisticated surrogate methods, such as the RBF, PRS, SVR, Kriging, and DKG [2-10] methods, are better. In any case, it is recommended to sample inside the region of $(\mu_X - 3\sigma_X,\ \mu_X + 3\sigma_X)$ for the random variable *X* if the distribution has an infinite domain.

4. Numerical Examples

Analytical examples and an engineering example are used to test the performance of the proposed variable screening method. Partial output variances are calculated to select important variables using the 1-D quadratic interpolation presented in Section 3.3. To use the variable screening method, five parameters: significance level α, false negative rate β, number of sample ns, initial number of repetition nr_0 and control parameter γ for threshold value, need to be decided by users. Smaller α and β are better choices because they result in less statistical errors in the variable screening method. However, when they are too small, very large nr could be required to maintain the error level specified by α and β in Eq. (23). Hence, 0.025 to 0.05 would be a reasonable choice for them. For nr_0 and ns, a small number could be chosen to reduce computational cost. However, a small value of nr_0 and ns will rapidly increase nr to maintain the error level. Hence, an appropriately large number should be used; they are set to be 50 or 100 in the numerical examples. The parameter γ in Eq. (22) is for user control of the threshold value that determines important variables. In the numerical example, γ is initially set to 1.0, and the variable screening procedure is performed. Then, the ratio of the sum of partial output variances of selected variables to that of all random variables, which is an estimation of the captured output variance, is checked. If the ratio is less than 85%, γ is lowered to achieve 85%. As explained before, the partial output variance is an approximation method, which is why the ratio of 85% may not mean that 85% of total output variance is actually captured in the selected variables. However, it would be a good estimation with affordable cost because it does not require many design of experiments (DoE) samples or full-dimensional surrogate models.

<u>4.1 Analytical Examples</u>

Hartmann 6-D and Dixon-Price 12-D are well-known analytical functions. They are high-dimensional as well as nonlinear, so they are tested for the variable screening method. Constant terms are added to the original functions to make both functions active (i.e. $G(\mathbf{X}) = 0$) at the mean point of the input random variables. Note that adding a constant term does not change the character of the functions. Input random variables have a variety of marginal distribution types and copula types, so the analytical examples can reveal the effects of different distribution types and correlations.

The functions are tested with three different methods. The first is the variable screening method. As mentioned before, parameters of $\alpha = \beta = 0.025$, $nr_0 = ns = 100$ and $\gamma = 1.0$, and 1-D surrogate model with quadratic interpolation are used. γ is initially set to 1.0 and lowered when necessary. The second is screening with accurate partial output variances using the analytical functions directly and 1,000,000 realizations of random variables. The calculated partial output variances are used as reference. When a performance measure is a linear function ($G = \Sigma\alpha_i X_i$) of the input random variables X_i's, the output variance is $\Sigma\alpha_i^2\sigma_{X_i}^2$, where $\sigma_{X_i}^2$ is variance X_i. Hence, the partial output variance can be linearly approximated as $\alpha_i^2\sigma_{X_i}^2$ with a design sensitivity (gradient) α_i and the input variance $\sigma_{X_i}^2$. Furthermore, an important variable might be selected based on the partial output variances calculated with the sensitivity-variance method, and it is applied to the analytic functions for comparison.

4.1.1 Hartmann 6-D

The first analytical example is the Hartmann 6-D, and a constant term is added as explained before. The analytical expression is shown as [37]

$$G(\mathbf{X}) = -\sum_{i=1}^{q} a_i \exp\left(-\sum_{j=1}^{m} b_{ij}(X_j - d_{ij})^2\right) + 3.3082 \tag{25}$$

where $0 \le X_i \le 1$, $m = 6$, $q = 4$ and

$$\mathbf{a} = \begin{bmatrix} 1.0 & 1.2 & 3.0 & 3.2 \end{bmatrix} \tag{26}$$

$$\mathbf{b} = \begin{bmatrix} 10.0 & 3.0 & 17.0 & 3.5 & 1.7 & 8.0 \\ 0.05 & 10.0 & 17.0 & 0.1 & 8.0 & 14.0 \\ 3.0 & 3.5 & 1.7 & 10.0 & 17.0 & 8.0 \\ 17.0 & 8.0 & 0.05 & 10.0 & 0.1 & 14.0 \end{bmatrix} \tag{27}$$

$$\mathbf{d} = \begin{bmatrix} 0.1312 & 0.1696 & 0.5569 & 0.0124 & 0.8283 & 0.5886 \\ 0.2329 & 0.4135 & 0.8307 & 0.3736 & 0.1004 & 0.9991 \\ 0.2348 & 0.1451 & 0.3522 & 0.2883 & 0.3047 & 0.6650 \\ 0.4047 & 0.8828 & 0.8732 & 0.5743 & 0.1091 & 0.0381 \end{bmatrix} \tag{28}$$

Information about input random variables is listed in Table 2. Input random variables have four different marginal distribution types: normal, lognormal, gamma, and Weibull. X_5 and X_6 are correlated with the Clayton copula and Kendall's tau of 0.5.

Table 2. Input Random Variables for Hartmann 6-D Example

Random Variable	Distribution Type	Mean	STDEV	Correlation (Copula)
X_1	Normal	0.20	0.009	-
X_2	Lognormal	0.15	0.007	-
X_3	Gamma	0.48	0.015	-
X_4	Weibull	0.28	0.014	-
X_5	Normal	0.31	0.014	Clayton
X_6	Weibull	0.68	0.013	$\tau = 0.5$

The result of variable screening is shown in Table 3. The design sensitivity (gradient) of the Hartmann 6-D in Eq. (25) at the input mean point is shown in the second column, and the third through fifth columns show partial output variances using the sensitivity-variance method, the variable screening method, and the accurate method, respectively. As the variable screening method calculates partial output variances nr times, the result is the mean value of the calculated partial output variances. In each method, important variables are marked with bold font. It can be seen that the variable screening method finds the same variables as the accurate method, whereas the sensitivity-variance method misses X_4 and X_5. The sixth and seventh columns are the ratios of partial output variances using the sensitivity-variance method and the variable screening method to the accurate partial output variances, respectively. It is evident that the sensitivity-variance method cannot estimate the partial output variances accurately, while the variable screening method does. Overall, the variable-screening method outperforms the

sensitivity-variance method. Hence, it is better to use at least quadratic approximation for the 1-D surrogate model to calculate partial output variances.

Table 3. Partial Output Variances of Hartman 6-D Example

Random Variable	Design Sensitivity	Sensitivity -Variance	Variable Screening	Accurate	Sens.-Var. /Accurate	Variable Scr. /Accurate
X_1	−5.88E−02	2.80E−07	2.24E−06	2.25E−06	12.4%	99.6%
X_2	−1.36E−02	9.04E−09	6.12E−07	6.17E−07	1.5%	99.2%
X_3	7.15E−02	1.15E−06	1.11E−05	1.07E−05	10.7%	103.7%
X_4	2.63E−01	1.35E−05	**8.88E−05**	**8.67E−05**	15.6%	102.4%
X_5	−1.36E−01	3.61E−06	**1.86E−04**	**1.86E−04**	1.9%	100.0%
X_6	1.17E+00	**2.29E−04**	**1.65E−04**	**1.66E−04**	138.0%	99.4%
Selected		2.29E−04	4.40E−04	4.39E−04	-	-
All		2.48E−04	4.54E−04	4.52E−04	-	-
Selected / All		92.3%	96.9%	97.1%	-	-
Criterion (μ_0)		4.13E−05	7.42E−05	7.54E−05	-	-

In Table 3, the bottom four rows show more information about each method. The first and second rows are the sums of partial output variances of selected variables and all variables, respectively. The third row shows the ratio of the first row to the second row. The last row is μ_0, which is the criterion used to select important variables. In the variable screening method, the important variables (X_4, X_5 and X_6) are determined to be larger than μ_0 by hypothesis testing. The sensitivity-variance method and the accurate method select important variables if the variable has partial output variance larger than μ_0. In the third row, the ratio for the variable screening method is larger than 85%, so γ is the initial value of 1.0, and equivalent μ_0 is applied for other methods. The sensitivity-variance method estimates that 92.3% of output variance is contained in X_6 only. This is a very poor estimation, as only 36.7% (=1.66E−04/4.52E−04) of output variance is captured in X_6 according to the result of the accurate method. On the other hand, the variable screening method estimates that 96.9% of output variance is contained in X_4, X_5 and X_6, and this is very accurate compared to the 97.1% determined by the accurate method.

However, the total and captured output variances in Table 3 are approximation using the partial output variances and the correlation term is not considered as explained in Section 3.1. Having the analytical expression of the Hartman 6-D example in Eq. (25), true total output variance induced by multiple input random variables can be calculated as well. The true total output variance is calculated using 1,000,000 realizations of all input random variables and the calculated value is 4.23E−04 as shown in Table 4. Recalling the approximated result in Table 3, the variable screening method (4.54E−04) and the accurate method (4.52E−04) well approximate the true total output variance, whereas the sensitivity-variance method (2.48E−04) is not able to do so. In Table 4, true captured output variance by the selected variables is also calculated. To calculate the true captured output variance, the realizations, which are generated to calculate the true total output variances, are used. Among them, the realizations of the screened-out variables are fixed at their mean values. Then, the variance of Hartman 6-D is calculated using the modified realizations. The true captured output variance in X_6 is 1.66E−04, which is the same as the partial output variance of X_6 found by the accurate method (see Table 3). Hence, the sensitivity-variance method captures only 39.2% (1.66E−04/4.23E−04) of the true total output variance in its selection X_6. This will lead a reliability

problem not to estimate the probability of failure correctly. By contrast, the captured output variance in X_4, X_5 and X_6 is 4.10E−04, and this is 96.9% (4.10E−04/4.23E−04) of the total output variance. This indicates that a reliability problem could be solved accurately utilizing X_4, X_5 and X_6. From this example, it can be seen that the variable screening method works as it is intended.

Table 4. True Total and Captured Output Variances of Hartman 6-D Example

	Sensitivity -Variance	Variable Screening & Accurate	True Total Output Variance
Selected Variables	X_6 (1-D)	X_4, X_5, X_6 (3-D)	All (6-D)
Captured Output Variance	1.66E−04	4.10E−04	4.23E−04
Ratio to Total Out. Var.	39.2%	96.9%	100.0%

4.1.2 Dixon-Price 12-D

The second analytical example is the Dixon-Price 12-D, and again a constant term is added to its original function. The analytical expression is shown as [38]

$$G(\mathbf{X}) = (X_1 - 1)^2 + \sum_{i=2}^{m} i(2X_i^2 - X_{i-1})^2 - 3.5575 \times 10^{-3} \qquad (29)$$

where $-10 \le x_i \le 10$, $i = 1, 2, \ldots, m$ and $m = 12$. Input random variables shown in Table 5 are used for the test. They have five different marginal distribution types of normal, lognormal, Weibull, Gumbel, and Gamma. X_1 and X_2 are correlated with the Frank copula and Kendall's tau of 0.7. Also, X_5 and X_6 are correlated with the FGM copula and Kendall's tau of 0.2.

Table 5. Input Random Variables for Dixon-Price 12-D Example

Random Variable	Distribution Type	Mean	STDEV	Correlation (Copula)
X_1	Normal	1.00	0.025	Frank
X_2	Normal	0.71	0.02	$\tau = 0.7$
X_3	Lognormal	0.59	0.02	-
X_4	Lognormal	0.55	0.02	-
X_5	Weibull	0.52	0.02	FGM
X_6	Weibull	0.51	0.02	$\tau = 0.2$
X_7	Gumbel	0.51	0.015	-
X_8	Gumbel	0.50	0.015	-
X_9	Normal	0.50	0.015	-
X_{10}	Normal	0.50	0.01	-
X_{11}	Gamma	0.50	0.01	-
X_{12}	Gamma	0.50	0.015	-

The test result of the Dixon-Price 12-D example is shown in Table 6, and selected variables in each method are marked with bold font. In this example, the value of γ is lowered to 0.7 to contain at least 85% of output variance in

the selected variables. And it is shown that 86.6% of output variance is estimated using the variable screening method in Table 6. Design sensitivities with respect to $X_9 \sim X_{12}$ are zero, and accordingly the partial output variances of $X_9 \sim X_{12}$ using the sensitivity-variance method are zero. Hence, the sensitivity-variance method misses X_9 and X_{12} even though they have large partial output variances. Moreover, the other partial output variances using the sensitivity-variance method have poor accuracy compared to the accurate method (see the sixth column in Table 6). Hence, X_2 and X_3 are selected instead of X_6 even though X_6 actually has larger partial output variance than X_2 and X_3. On the contrary, the variable screening method reasonably estimates partial output variances and correctly identifies important variables compared to the accurate method. Therefore, it is confirmed that at least quadratic approximation is needed for the 1-D surrogate model to calculate partial output variances.

Table 6. Partial Output Variances of Dixon-Price 12-D Example

Random Variable	Design Sensitivity	Sensitivity -Variance	Variable Screening	Accurate	Sens.-Var. /Accurate	Variable Scr. /Accurate
X_1	−3.28E−02	6.72E−07	7.95E−06	7.71E−06	8.7%	103.1%
X_2	1.76E−01	**1.24E−05**	1.41E−04	1.35E−04	9.2%	104.4%
X_3	−3.15E−01	**3.98E−05**	1.56E−04	1.62E−04	24.6%	96.3%
X_4	3.56E−01	**5.07E−05**	**2.95E−04**	**2.81E−04**	18.0%	105.0%
X_5	−1.94E−01	**1.50E−05**	4.98E−04	4.45E−04	3.4%	111.9%
X_6	−1.38E−01	7.61E−06	**6.22E−04**	**5.51E−04**	1.4%	112.9%
X_7	4.51E−01	**4.58E−05**	**5.27E−04**	**5.57E−04**	8.2%	94.6%
X_8	−3.20E−01	**2.30E−05**	**3.10E−04**	**3.55E−04**	6.5%	87.3%
X_9	0.00E+00	0.00E+00	**2.20E−04**	**2.15E−04**	0.0%	102.3%
X_{10}	0.00E+00	0.00E+00	5.31E−05	5.25E−05	0.0%	101.1%
X_{11}	0.00E+00	0.00E+00	6.42E−05	6.32E−05	0.0%	101.6%
X_{12}	0.00E+00	0.00E+00	**2.45E−04**	**2.39E−04**	0.0%	102.5%
Selected		1.87E−04	2.72E−03	2.64E−03	-	-
All		1.95E−04	3.14E−03	3.06E−03	-	-
Selected / All		95.9%	86.6%	86.3%	-	-
Criterion (μ_0)		1.14E−05	1.68E−04	1.79E−04	-	-

Using Eq. (29), the true total and captured output variances of Dixon-Price 12-D example are calculated as shown in Table 7. In Table 6, the variable screening (3.14E−03) and the accurate methods (3.06E−03) reasonably approximate the true total output variance (3.30E−03 in Table 7), while the sensitivity-variance method (1.95E−04) cannot. In Table 7, the true captured output variance by selected variables using the sensitivity-variance method is

only 2.15E−03, which is 65.2% of the true total output variance. By contrast, the output variance of 2.80E−03 is contained in $X_4 \sim X_9$ and X_{12}, which indicates that 84.8% of the true total output variance is captured. Hence, it is verified that the variable screening method correctly finds the important variables of the Dixon-Price 12-D example. Through analytical examples, it is shown that the partial output variance is a well-performing measure for variable screening purposes, and the proposed variable screening method successfully finds important variables as it is intended.

Table 7. True Total and Captured Output Variances of Dixon-Price 12-D Example

	Sensitivity -Variance	Variable Screening & Accurate	True Total Output Variance
Selected Variables	$X_2 \sim X_5$, X_7, X_8 (6-D)	$X_4 \sim X_9$, X_{12} (7-D)	All (12-D)
Captured Output Variance	2.15E−03	2.80E−03	3.30E−03
Ratio to Total Out. Var.	65.2%	84.8%	100.0%

4.2 Engineering Example

A car noise, vibration, and harshness (NVH) and crash safety problem is considered to demonstrate the performance and efficiency of the proposed method. The problem includes full frontal impact, 40% offset frontal impact, and NVH as constraints. There are a total of 11 performance measures as shown in Table 8: nine safety measures and two NVH measures.

Table 8. Performance Measure Description

Mode		Function	Value	Feasibility decision
Safety	Full frontal impact	G_1	Chest G	$\leq Baseline_i$
		G_2	Crush displacement	
	40% offset impact	G_3	Brake pedal	
		G_4	Footrest	
		G_5	Left toepan	
		G_6	Center toepan	
		G_7	Right toepan	
		G_8	Left IP	
		G_9	Right IP	
NVH		G_{10}	Torsion mode	
		G_{11}	Vertical bending mode	

In this example, it is assumed that the only source of uncertainty is the thickness of the body plates. The 44 random variables shown in Table 9 are used to represent the thicknesses. All random variables follow normal distribution and are statistically independent. The design variable vector **d** is the mean vector of the 44 random variables, and there is no random parameter in this example. Among those random variables, six random variables (X_1 to X_5 and X_8) are common variables for both safety and NVH measures, two (X_6 and X_7) are variables only for safety, and the other 36 random variables are only for NVH measures.

Table 9. Input Random Variables

RVs	Dist. Type	**d**	STDV	$\mathbf{d}^L$	$\mathbf{d}^U$	RVs	Dist. Type	**d**	STDV	$\mathbf{d}^L$	$\mathbf{d}^U$
X_1	Normal	1.9	0.05	1.5	2.3	XN_1	Normal	0.9	0.03	0.7	1.1

X_2	Normal	1.91	0.05	1.5	2.3	XN_2	Normal	1.1	0.03	0.8	1.4
X_3	Normal	2.51	0.06	2.0	3.0	XN_3	Normal	1.55	0.05	1.2	1.9
X_4	Normal	2.4	0.06	1.9	2.9	XN_4	Normal	0.9	0.03	0.7	1.1
X_5	Normal	2.55	0.06	2.0	3.1	XN_5	Normal	1.5	0.03	1.2	1.8
X_6	Normal	2.25	0.06	1.8	2.7	XN_6	Normal	1.2	0.03	0.9	1.5
X_7	Normal	2.25	0.06	1.8	2.7	XN_7	Normal	1.1	0.03	0.8	1.4
X_8	Normal	1.5	0.03	1.2	1.8	XN_8	Normal	1.52	0.05	1.2	1.9
X_{10}	Normal	1.28	0.03	0.9	1.6	XN_9	Normal	0.8	0.03	0.6	1.0
X_{11}	Normal	1.4	0.03	1.0	1.8	XN_{10}	Normal	0.8	0.03	0.6	1.0
X_{12}	Normal	1.1	0.03	0.8	1.4	XN_{11}	Normal	1.2	0.03	0.9	1.5
X_{13}	Normal	2.2	0.06	1.7	2.7	XN_{12}	Normal	0.75	0.03	0.6	0.9
X_{14}	Normal	1.5	0.03	1.2	1.8	XN_{13}	Normal	0.75	0.03	0.6	0.9
X_{15}	Normal	1.25	0.03	0.9	1.6	XN_{14}	Normal	0.75	0.03	0.6	0.9
X_{16}	Normal	2.5	0.06	2.0	3.0	XN_{15}	Normal	1.0	0.03	0.8	1.2
X_{17}	Normal	2.0	0.05	1.5	2.5	XN_{16}	Normal	1.14	0.03	0.9	1.4
X_{18}	Normal	1.4	0.03	1.1	1.7	XN_{17}	Normal	1.2	0.03	0.9	1.5
X_{20}	Normal	1.22	0.03	0.9	1.5	XN_{18}	Normal	1.4	0.03	1.1	1.7
X_{23}	Normal	0.75	0.03	0.6	1.0	XN_{19}	Normal	1.2	0.03	0.9	1.5
X_{24}	Normal	1.9	0.05	1.5	2.3	XN_{20}	Normal	1.4	0.03	1.1	1.7
X_{25}	Normal	0.65	0.03	0.5	0.8	XN_{21}	Normal	2.13	0.06	1.7	2.6
X_{26}	Normal	0.85	0.03	0.6	1.1						
X_{27}	Normal	0.85	0.03	0.6	1.1						

This problem requires three and a half hours for the impact dynamic analysis for crash safety and the modal analysis for NVH. Thus, the actual analysis takes too much time to test the proposed method thoroughly. Ford Motor Company provided full-dimensional global (considering the entire design domain) surrogate models so that we can use them to demonstrate the proposed method of variable screening. The full-dimensional surrogate models may not be accurate, since 44-D is too high to create accurate surrogate models, especially for RBDO. However, to test the proposed method of variable screening, the responses from the 44-D global surrogate models are treated as true responses in this paper. The maximum dimension at which accurate surrogate models can be generated depends on the computational power and nonlinearity of a given problem. In this paper, the DKG method [10] is used to generate an accurate surrogate model using the Iowa Reliability-Based Design Optimization (I-RBDO) code [39], and 18-D is targeted as the maximum degrees of freedom of DKG models. The I-RBDO code is also used to carry out RBDO in this paper.

4.2.1 Variable Screening

At the baseline design **d**, which is the initial design as shown in Table 9, all 11 performance measures in Table 8 are active. That is, the value of every performance measure at the baseline design is the same as the baseline values, with $G_i=Baseline_i$, i=1~11. Therefore, the proposed variable screening method is performed for the problem at the baseline design. Parameters *ns*, nr_0, α, β, and γ are set as 50, 50, 0.05, 0.05, and 1.0, respectively. Four hundred eighty-four (44 design variables × 11 performance measures) 1-D surrogate models with quadratic interpolation are generated using 89 DoE samples (i.e., simulation samples). It is noted that 11 values of performance measures are obtained from one analysis. The results of partial output variances $\bar{v}_i$ are listed in Table 10 and Table 11 for every performance measure. The partial output variances of the important variables for each performance measure are marked with bold font. It is noted that only partial output variances of X_1~X_8 are listed in Table 10 since G_1~G_9 are only a function of X_1~X_8 as the variable screening method identified the partial output

variances to be zero for other random variables. In Table 11, X_6 and X_7 have zero partial output variances as G_{10} and G_{11} are not functions of X_6 and X_7. In the last three rows of Table 10 and Table 11, the sums of partial output variances of selected variables, the sums of all partial output variances, and their ratios are listed. As explained before, this is the estimated ratio between the captured output variance in selected variables to the total output variance. It is estimated that a minimum of 90.3% of the total output variance is captured in the selected variables. In total, 14 random variables: X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_{10}, X_{20}, X_{23}, X_{25}, X_{26}, and XN_1, are selected as important variables. Accordingly, 14 design variables, which are the means of the selected random variables, are considered as important design variables.

Table 10. Partial Output Variances $\bar{v}_i$ ($G_1 \sim G_9$)

RVs	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8	G_9
X_1	**2.37E−02**	**3.75E+00**	**4.06E+01**	**1.11E+01**	**1.94E+00**	**1.73E+00**	**6.21E−01**	**3.45E−01**	**5.75E−01**
X_2	**1.88E−02**	**1.01E+00**	3.55E−01	**1.86E+00**	**2.07E+00**	**2.06E+00**	**1.09E+00**	**4.03E−01**	**2.69E−01**
X_3	5.08E−05	1.76E−02	**1.05E+01**	**9.74E+00**	**5.95E+00**	**4.38E+00**	**2.00E+00**	**2.37E−01**	2.24E−02
X_4	1.66E−04	9.27E−02	**3.91E+00**	**1.79E+00**	2.65E−01	6.65E−02	1.73E−03	4.78E−03	9.65E−03
X_5	1.12E−04	1.37E−04	**2.74E+00**	**2.23E+00**	**4.90E−01**	**9.11E−01**	**9.30E−01**	**1.79E−01**	**7.94E−02**
X_6	4.58E−05	**2.77E+00**	1.64E−01	8.14E−02	**8.18E−01**	1.60E−01	**1.70E−01**	3.95E−02	3.27E−02
X_7	**2.32E−03**	1.34E−01	1.91E−01	1.50E−01	**2.75E+00**	**1.71E+00**	**5.66E−01**	**1.10E−01**	**2.77E−01**
X_8	**1.23E−03**	6.28E−02	4.39E−02	3.02E−01	**5.35E−01**	**5.45E−01**	**3.27E−01**	6.25E−02	4.87E−02
Selected	4.61E−02	7.53E+00	5.78E+01	2.67E+01	1.46E+01	1.13E+01	5.70E+00	1.38E+00	1.28E+00
All	4.64E−02	7.84E+00	5.85E+01	2.73E+01	1.48E+01	1.16E+01	5.71E+00	1.38E+00	1.31E+00
Selec./All	99.4%	96.0%	98.8%	97.8%	98.6%	97.4%	99.8%	100.0%	97.7%

Table 11. Partial Output Variances $\bar{v}_i$ (G_{10} and G_{11})

RVs	G_{10}	G_{11}	RVs	G_{10}	G_{11}	RVs	G_{10}	G_{11}
X_1	**1.30E−03**	**1.48E−03**	X_{17}	4.24E−07	2.01E−04	XN_8	2.06E−05	5.45E−05
X_2	4.39E−05	3.44E−05	X_{18}	9.71E−06	3.33E−05	XN_9	1.61E−06	1.89E−05
X_3	**8.95E−04**	3.05E−04	X_{20}	**2.21E−03**	**1.84E−03**	XN_{10}	5.11E−06	1.83E−06
X_4	8.74E−05	2.63E−05	X_{23}	**2.67E−03**	**1.13E−03**	XN_{11}	1.56E−07	1.12E−07
X_5	3.06E−04	4.23E−05	X_{24}	2.35E−05	6.89E−05	XN_{12}	1.83E−06	3.10E−05
X_6	0	0	X_{25}	**4.85E−04**	**7.31E−03**	XN_{13}	1.69E−07	8.19E−07
X_7	0	0	X_{26}	**4.59E−03**	**1.06E−02**	XN_{14}	9.08E−08	3.75E−05
X_8	**5.56E−04**	1.89E−04	X_{27}	3.43E−05	3.76E−04	XN_{15}	1.84E−06	1.66E−05
X_{10}	**5.49E−04**	3.87E−04	XN_1	9.33E−05	**6.38E−03**	XN_{16}	4.35E−08	2.88E−08
X_{11}	3.32E−05	2.05E−04	XN_2	4.49E−07	7.48E−05	XN_{17}	3.24E−07	1.84E−06
X_{12}	9.35E−05	3.98E−04	XN_3	9.59E−08	2.00E−05	XN_{18}	1.04E−07	1.18E−07
X_{13}	7.23E−06	3.82E−04	XN_4	6.04E−08	8.36E−05	XN_{19}	3.03E−08	3.37E−07
X_{14}	5.76E−07	1.17E−05	XN_5	2.60E−07	1.42E−07	XN_{20}	2.06E−07	2.98E−06
X_{15}	1.16E−07	1.13E−05	XN_6	1.29E−07	2.22E−07	XN_{21}	2.44E−07	9.41E−06
X_{16}	9.14E−08	1.63E−06	XN_7	9.46E−08	7.94E−06			
Selected							1.33E−02	2.87E−02
All							1.40E−02	3.18E−02
Selec./All							95.0%	90.3%

Sensitivity-variance introduced in Section 4.1 is applied to performance measure G_5, and the result is shown in Table 12. Selected random variables for G_5 are $X_1 \sim X_3$ and $X_5 \sim X_8$ out of $X_1 \sim X_8$. Among $X_1 \sim X_8$, variables $X_3 \sim X_7$ have the largest standard deviation of 0.06. However, X_4 is not selected among them because it has small design sensitivity compared to others. Here, the design sensitivities are calculated at the design point using the forward

finite difference method (FDM) with 0.1% perturbation. By contrast, X_8 is selected as an important variable even though it has smallest standard deviation of 0.03. Again, this is because it has relatively large sensitivity and induces large output variances.

Table 12. Result of Sensitivity-Variance Method in G_5

RVs	STDEV (σ_{X_i})	Design Sensitivity (α_i)	Partial Output Variance ($\alpha_i^2 \sigma_{X_i}^2$)
X_1	0.05	27.9105	1.95E+00
X_2	0.05	29.2723	2.14E+00
X_3	0.06	−40.7490	5.98E+00
X_4	0.06	8.7917	2.78E−01
X_5	0.06	−11.6078	4.85E−01
X_6	0.06	14.1289	7.19E−01
X_7	0.06	27.8533	2.79E+00
X_8	0.03	24.3000	5.31E−01

Interestingly, the partial output variances using the sensitivity-variance method of G_5 shown in Table 12 are close to the result shown in the sixth column of Table 10. In fact, the same variables as those in the variable screening method will be selected by using the sensitivity-variance method throughout all 11 constraints. However, the sensitivity-variance method has the possibility of choosing undesirable variables as shown in the analytical examples in Section 4.1. In Fig. 3, the shape of G_5 when each X_i is random is shown. It is easily anticipated that the design sensitivity of G_5 with respect to X_i could be very small or even zero so that the sensitivity-variance method may provide inaccurate partial output variance; this can be prevented if the variable screening method is used.
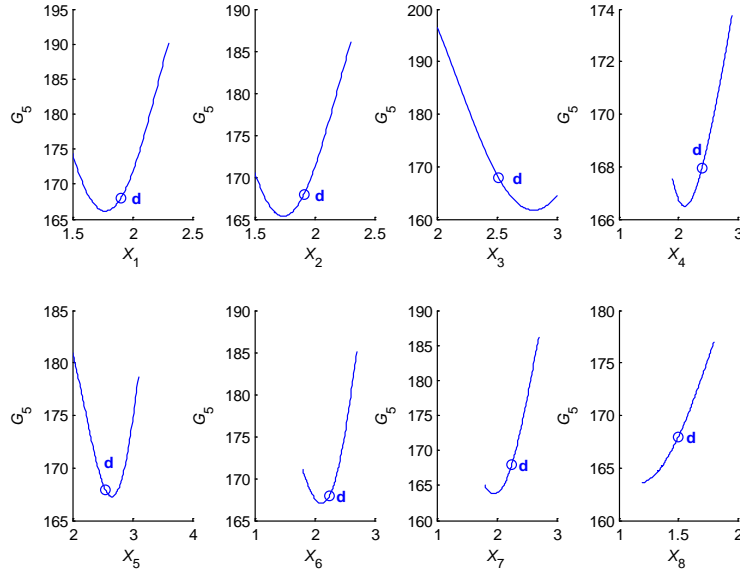

Figure 3. Shape of G_5 when Each X_i is Random

The sensitivity-variance method requires accurate design sensitivity. In practical engineering problems, design sensitivity might be calculated using FDM. To use FDM, a user determines perturbation method (forward, backward or central) and perturbation size, and the result of the sensitivity-variance could depend on the user's choice. In Table 13, partial output variance of X_2 in G_6 at DDO optimum design with various methods is shown. It can be seen that X_2 is not selected as an important variable when design sensitivity is calculated using forward FDM with 1% perturbation. The partial output variance is only 56.6% of that found using the accurate method with 100,000 realizations of X_2. To obtain more accurate design sensitivity with forward FDM, small perturbation is required as shown in Table 13. However, small perturbation does not always provide accurate design sensitivity, and determining appropriate perturbation size would require extra DoE samples. Central FDM provides more accurate design sensitivity, and it is insensitive to perturbation size. However, the partial output variance with central FDM sensitivity shows at most 76.7% accuracy compared to the accurate method. It is noted that the variable screening method would not require perturbation size determination. Moreover, a user can perform the proposed variable screening method using only one more DoE sample than the sensitivity-variance method with central FDM.

Table 13. Partial Output Variance of X_2 in G_6 at DDO Optimum Design

Method	Perturbation	Selection	Design Sensitivity	Partial Output Variance	Accuracy
Sensitivity-Variance	Forward, 1%	No	−5.83E+00	8.49E−02	56.6%
	Forward, 0.5%	Yes	−6.30E+00	9.94E−02	66.3%
	Forward, 0.1%	Yes	−6.68E+00	1.12E−01	74.7%
	Central, 1%	Yes	−6.79E+00	1.15E−01	76.7%
	Central, 0.5%	Yes	−6.78E+00	1.15E−01	76.7%
	Central, 0.1%	Yes	−6.78E+00	1.15E−01	76.7%
Variable Screening	-	Yes	-	1.51E−01	100.7%
Accurate	-	Yes	-	1.50E−01	100.0%

Since we have 44-D global surrogate models for this example, GSA can be carried out to verify effectiveness of the proposed method. Among various GSA methods, the global sensitivity index method, which can identify the global effect of the variables of interest on the output, is used here. The main strength of the global sensitivity index method is that it can find interactions between all variables (not statistical correlation between random variables). All random variables are assumed to follow uniform distribution in their corresponding design domain of $\mathbf{d}^L$ and $\mathbf{d}^U$, and global sensitivity indices are calculated using the Monte Carlo simulation (MCS) method with 1 million MCS

samples [31]. There are many global sensitivity indices in this 44-D problem; the total sensitivity index S_i^{tot} is used for variable ranking and screening. The total sensitivity index S_i^{tot} is a "total influence of ith random variable" to the output. That is, it indicates the main effect plus interactions of the ith random variable with other random variables [22]. The results are listed in Table 14 and Table 15. To identify important random variables, the mean value of S_i^{tot} is calculated for each constraint and the random variable, which yields larger S_i^{tot} than the mean value, is selected as an important variable and marked in bold font in these tables. In Table 14, only S_i^{tot} for $X_1 \sim X_8$ are listed as S_i^{tot} for other variables are zero. Also, the sum of S_i^{tot} for the selected random variables, the sum of S_i^{tot} for all random variables, and their ratios are listed in the last three rows, respectively.

Table 14. Global Sensitivity Indices S_i^{tot} ($G_1 \sim G_9$)

RVs	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8	G_9
X_1	**4.16E−01**	**3.84E−01**	**5.23E−01**	**2.68E−01**	**1.74E−01**	**1.89E−01**	**2.07E−01**	**2.56E−01**	**4.35E−01**
X_2	**3.46E−01**	**1.18E−01**	**1.04E−01**	**1.88E−01**	**1.98E−01**	**2.09E−01**	**1.66E−01**	**2.62E−01**	**2.07E−01**
X_3	7.83E−03	5.75E−03	**3.28E−01**	**5.07E−01**	**5.18E−01**	**4.97E−01**	**4.92E−01**	**2.69E−01**	4.96E−02
X_4	1.89E−02	**2.23E−02**	**7.92E−02**	**1.34E−01**	**1.25E−01**	**1.38E−01**	**1.42E−01**	7.38E−02	2.87E−02
X_5	7.50E−03	−7.22E−04	**1.40E−01**	**2.10E−01**	**1.53E−01**	**1.76E−01**	**1.78E−01**	**1.90E−01**	9.10E−02
X_6	**5.65E−02**	**2.67E−01**	2.08E−02	**5.01E−02**	**1.11E−01**	**1.15E−01**	**1.67E−01**	5.13E−02	2.74E−02
X_7	**8.13E−02**	5.65E−03	1.77E−02	**4.52E−02**	**1.59E−01**	**1.60E−01**	**1.76E−01**	6.73E−02	**1.73E−01**
X_8	**3.23E−02**	**2.06E−02**	4.02E−03	2.33E−02	**3.82E−02**	**4.09E−02**	5.21E−02	7.13E−02	6.14E−02
Selected	9.32E−01	8.12E−01	1.17E+00	1.40E+00	1.48E+00	1.52E+00	1.58E+00	1.24E+00	1.07E+00
All	9.66E−01	8.23E−01	1.22E+00	1.43E+00	1.48E+00	1.52E+00	1.58E+00	1.24E+00	1.07E+00
Selec./All	96.5%	98.7%	95.9%	97.9%	100.0%	100.0%	100.0%	100.0%	100.0%

Table 15. Global Sensitivity Indices S_i^{tot} (G_{10} and G_{11})

RVs	G_{10}	G_{11}	RVs	G_{10}	G_{11}	RVs	G_{10}	G_{11}
X_1	**1.05E−01**	**7.09E−02**	X_{17}	3.30E−04	1.46E−02	XN_8	−8.85E−04	−1.34E−03
X_2	1.01E−03	2.38E−03	X_{18}	−4.77E−04	−2.50E−04	XN_9	8.54E−04	−1.71E−03
X_3	**5.72E−02**	1.01E−02	X_{20}	**2.07E−01**	**9.17E−02**	XN_{10}	1.28E−04	−1.07E−04
X_4	8.66E−03	2.65E−03	X_{23}	**1.34E−01**	**3.57E−02**	XN_{11}	−2.72E−04	−2.85E−05
X_5	**3.29E−02**	5.69E−03	X_{24}	−1.60E−03	−8.39E−04	XN_{12}	−1.51E−04	−1.94E−04
X_6	0	0	X_{25}	1.12E−02	**9.79E−02**	XN_{13}	−5.24E−05	1.59E−04
X_7	0	0	X_{26}	**2.96E−01**	**3.80E−01**	XN_{14}	−3.88E−05	−6.91E−05
X_8	**4.73E−02**	5.89E−03	X_{27}	6.04E−03	**2.38E−02**	XN_{15}	1.20E−04	4.59E−04
X_{10}	**5.98E−02**	1.99E−02	XN_1	5.62E−03	**1.65E−01**	XN_{16}	8.87E−05	2.85E−04
X_{11}	3.86E−03	1.67E−02	XN_2	−1.11E−04	2.31E−03	XN_{17}	−6.09E−05	−3.61E−04
X_{12}	1.34E−02	**2.88E−02**	XN_3	1.13E−04	1.33E−03	XN_{18}	−1.12E−05	−9.18E−05
X_{13}	3.20E−04	1.50E−02	XN_4	1.54E−04	−2.50E−03	XN_{19}	1.10E−04	1.23E−03
X_{14}	2.24E−04	−4.65E−04	XN_5	−2.96E−04	9.10E−04	XN_{20}	−1.67E−04	−8.38E−04
X_{15}	−1.33E−04	1.36E−03	XN_6	2.34E−04	−2.32E−05	XN_{21}	6.77E−05	−6.44E−04
X_{16}	−1.94E−04	2.42E−03	XN_7	−1.61E−05	−3.27E−04			
Selected							9.39E−01	8.94E−01
All							9.87E−01	9.87E−01
Selec./All							95.1%	90.6%

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Using the global sensitivity index method, 16 random variables are selected as shown in Table 16. Those 16 random variables include all 14 random variables selected using the proposed method as shown in Table 16. Moreover, if we set a limit of 14 random variables to be selected, X_{12} and X_{27} will not be selected as they have least S_i^{tot} among the selected variables for G_{11} as shown in Table 15. Thus, the 14 random variables selected by both methods are identical. The ratio between the sensitivity indices of selected variables and all variables has no physical meaning. However, it is an indicator that shows how much variance is captured by the selected random variables. The results are quite similar to those of the proposed method as shown in Tables 10 and 11 and Tables 14 and 15, respectively. Hence, it is demonstrated that the proposed variable screening method is quite effective even though it does not require global surrogate models unlike the global sensitivity index method.

Table 16. Selected Random Variables

Method	Selected Variables
Proposed Method	$X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_{10}, X_{20},$ $X_{23}, X_{25}, X_{26}, XN_1$ (14 RVs)
Global Sensitivity Index	$X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_{10}, X_{12},$ $X_{20}, X_{23}, X_{25}, X_{26}, X_{27}, XN_1$ (16 RVs)

4.2.2 Reliability-based Design Optimization

For this example, RBDO is formulated as

minimize   Weight (**d**)

$$\text{subject to} \quad \begin{array}{l} P\left[G_i(\mathbf{X}) > \text{Baseline}_i\right] \leq 10\%, \quad i = 1,\ldots,11 \\ \mathbf{d}^L \leq \mathbf{d} \leq \mathbf{d}^U, \quad \mathbf{d} \in \square^{\,NDV}, \quad \text{and} \quad \mathbf{X} \in \square^{\,NRV} \end{array} \tag{30}$$

For a comparison study, we considered three cases: (1) a set of 14 random variables are selected based on experience without using the proposed variable screening method, (2) another set of 14 random variables are selected using the proposed variable screening method as shown in Section 4.2.1, and (3) in addition to the 14 random variables selected in case 2, four more random variables are selected using the cost function sensitivity for a total of 18 design variables to test the effectiveness of the proposed variable screening method and accuracy of the I-RBDO code. The selected design variables are listed in Table 17.

Table 17. Selected Random Variables for RBDO

Cases	Common Selection	Different Selection
(1) Based on Experience		XN_9, XN_{10}, XN_{11}
(2) Variable Screening	$X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8,$ X_{20}, X_{25}, XN_1 (11 RVs)	X_{10}, X_{23}, X_{26}
(3) Variable Screening + Cost Function		X_{10}, X_{23}, X_{26} $+ XN_4, XN_9, XN_{10}, XN_{11}$

Because the cost function, which is weight in this problem, is a function of design variables **d**, not random variables **X**, the function is deterministic. Therefore, the design sensitivity of the cost function with respect to the design variable is calculated by FDM, and the four design variables (and related random variables XN_4, XN_9, $XN_{10,}$

and XN_{11}) that show the largest sensitivity among the unselected design variables are chosen. Then, RBDO is carried out with three sets of selected random variables. The optimum design results are summarized in Table 18. The bold font indicates chosen design variables, and others are fixed at the baseline design value. Also, probabilities of failure, cost function values, and design iteration details are listed in Table 19. All RBDOs are carried out using I-RBDO code with 500,000 MCS samples [10]. For the three cases, I-RBDO generates DKG surrogate models, using the sample responses obtained from the 44-D global surrogate models, which are treated as true responses, for RBDO. Since I-RBDO is able to carry out RBDO using the surrogate models generated by other methods (i.e., the surrogate models generated by Ford in this example) without having information on how the surrogate models are generated, the full-dimensional RBDO is performed as well using the 44-D surrogate models. The 44-D RBDO result is treated as the true RBDO optimum and used for the purpose of validation of RBDO results obtained for the three reduced-dimensional cases.

Indeed, the optimum design values for $d_1{\sim}d_8$ are very close to the full-dimensional 44-D case, as shown in Table 18, which shows that the DKG surrogate models generated in the I-RBDO code are accurate. In the three cases, the random variables $X_1{\sim}X_8$ are selected because they have large partial output variance for performance measures $G_1{\sim}G_9$. This means that they contribute a large portion of the output variance. Hence, finding optimum values for them is the most effective way to reduce the probabilities of failure of $G_1{\sim}G_9$. Similarly, d_{22} (corresponding to X_{26}) moves to the upper bound of 1.1 when it is selected because it has the largest partial output variances of G_{10} and G_{11}. The design variables d_{27}, d_{32}, $d_{33,}$ and d_{34} (corresponding to XN_4, XN_9, $XN_{10,}$ and XN_{11}), which are selected by the sensitivity of the cost function, move to their lower bounds of 0.7, 0.6, 0.6, and 0.9, respectively, to reduce the cost function without significantly affecting the reliability of the optimum design. On the other hand, even though some design variables are selected due to the partial output variances of some constraints, they move to their lower bounds. For example, d_{24} (corresponding to XN_1) moves to the lower bound of 0.7 because it has the largest sensitivity for the cost function, even though it has the third-largest partial output variance of G_{11}. That is, via the trade-offs in the optimization process, it is moved to the lower bound to minimize the cost function rather than to reduce the probability of failure.

In Table 19, the number of black box calls is listed. I-RBDO can deal with multiple samples simultaneously to create a surrogate model to utilize parallel computing. That is, one black-box call requests computational simulations at a number of sampling points; thus, the black box call indicates the clock time required for analyses for RBDO. Therefore, the number of black box calls represents actual computational cost more realistically than the number of CAEs when parallel computing is used. In this example, five samples are added to the DKG model at a time; therefore, the total number of CAEs is roughly five times the number of black box calls.

To verify once again that the surrogate model generated by DKG using I-RBDO is accurate, the same three cases are performed using I-RBDO by using responses from 44-D global surrogate model directly while fixing screened-out variables at their baseline design points. As shown in Table 20, the optimums found using DKG and 44-D global surrogate models are very close to each other. Hence, it is confirmed that DKG in I-RBDO generated an accurate surrogate model. Moreover, it is also verified that RBDO can be conducted based on an accurate surrogate model even for a moderately large-dimensional problem (14 and 18 dimensions).

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For the three reduced-dimensional cases, the probabilities of failure are calculated using only selected variables as random variables, since the other design variables are treated as deterministic as explained in Section 2.2 with fixed values at the baseline design. As shown in Table 19, it is noted that, in all cases, the target design constraints of 10% probability failure are closely satisfied as expected at these optimum designs since the RBDO considers only the selected variables as random variables. On the other hand, to check correct reliabilities, reliability analyses are carried out at these optimum designs treating all variables as random for the 44-D surrogate models and MCS with 1 million samples, as shown in Table 21. It is noted that probabilities of failure of full-dimensional optimum in Table 19 and Table 21 are different even though the discrepancy is negligible. Theoretically, they should be the same; however, they are not equal because different numbers of MCS samples (500,000 and 1 million) are used and MCS error is induced. At the baseline design, all constraints have approximately 50% of probability of failure, and this is reasonable because all constraints are active at the baseline design. However, they are not exactly 50% because the constraint functions are nonlinear. Probabilistic constraint results corresponding to $G_1 \sim G_9$ are active or feasible in both Table 19 and Table 21. Due to the fact that G_1 to G_9 are functions of X_1 to X_8, and all of them are selected as important variables, the RBDO result with reduced-dimension and the reliability analysis result with full-dimension are very close to each other, considering MCS errors. The constraint G_{10} shows inactive results regardless of which selection design variable set is used.

All probabilities of failure for the constraint G_{11} in Table 19 satisfy the target probability of failure 10%, which makes it obvious that these are the reliability analyses results of reduced-dimensional problems. However, full-dimensional reliability analyses at optimum designs show quite different values as shown in Table 21. Selection based on experience shows a 17.70% probability of failure, which violates the target reliability significantly. The variables selected based on experience contain only 55.40% of the total output variance of G_{11}. Hence, it cannot find any safe design once dimension is reduced. On the other hand, the probabilities of failure for the proposed variable screening method and also the one considering cost function are close to the target probability of failure. The selected variables contain 93.55% and 93.88% of the total output variance of G_{11} for Case 2 and Case 3, respectively. Hence it can find a correct optimum even with reduced dimension.

Table 18. RBDO Optimum Design

Design Variables	Corresponding RVs	Baseline Design	Based on Experience	Variable Screening	Variable Screening + Cost Function	Full Dimension
d_1	X_1	1.9	**1.8343**	**1.8366**	**1.8336**	**1.8491**
d_2	X_2	1.91	**2.1810**	**2.1804**	**2.1806**	**2.1692**
d_3	X_3	2.51	**2.8528**	**2.8561**	**2.8540**	**2.8803**
d_4	X_4	2.4	**1.9817**	**1.9810**	**1.9856**	**1.9507**
d_5	X_5	2.55	**2.7195**	**2.7228**	**2.7261**	**2.7296**
d_6	X_6	2.25	**2.2543**	**2.2497**	**2.2558**	**2.2408**
d_7	X_7	2.25	**2.3199**	**2.3185**	**2.3207**	**2.3284**
d_8	X_8	1.5	**1.7904**	**1.7966**	**1.7860**	**1.8**
d_9	X_{10}	1.28	1.28	**0.9**	**0.9**	**1.5336**
d_{10}	X_{11}	1.4	1.4	1.4	1.4	**1.0038**
d_{11}	X_{12}	1.1	1.1	1.1	1.1	**0.9868**
d_{12}	X_{13}	2.2	2.2	2.2	2.2	**1.7006**
d_{13}	X_{14}	1.5	1.5	1.5	1.5	**1.2**
d_{14}	X_{15}	1.25	1.25	1.25	1.25	**0.9**
d_{15}	X_{16}	2.5	2.5	2.5	2.5	**2.0**

d_{16}	X_{17}	2.0	2.0	2.0	2.0	**1.5033**
d_{17}	X_{18}	1.4	1.4	1.4	1.4	**1.1**
d_{18}	$\boldsymbol{X_{20}}$	1.22	**0.9**	**0.9**	**0.9**	**0.9**
d_{19}	$\boldsymbol{X_{23}}$	0.75	0.75	**0.6**	**0.6**	**0.6**
d_{20}	X_{24}	1.9	1.9	1.9	1.9	**1.5**
d_{21}	$\boldsymbol{X_{25}}$	0.65	**0.6897**	**0.5429**	**0.5871**	**0.8**
d_{22}	$\boldsymbol{X_{26}}$	0.85	0.85	**1.1**	**1.1**	**1.1**
d_{23}	X_{27}	0.85	0.85	0.85	0.85	**1.1**
d_{24}	$\boldsymbol{XN_1}$	0.9	**0.7**	**0.7**	**0.7**	**0.7**
d_{25}	XN_2	1.1	1.1	1.1	1.1	**0.8**
d_{26}	XN_3	1.55	1.55	1.55	1.55	**1.2039**
d_{27}	$\boldsymbol{XN_4}$	0.9	0.9	0.9	**0.7**	**0.7**
d_{28}	XN_5	1.5	1.5	1.5	1.5	**1.2**
d_{29}	XN_6	1.2	1.2	1.2	1.2	**0.9**
d_{30}	XN_7	1.1	1.1	1.1	1.1	**0.8**
d_{31}	XN_8	1.52	1.52	1.52	1.52	**1.6925**
d_{32}	$\boldsymbol{XN_9}$	0.8	**0.6**	0.8	**0.6**	**0.6**
d_{33}	$\boldsymbol{XN_{10}}$	0.8	**0.6**	0.8	**0.6**	**0.6**
d_{34}	$\boldsymbol{XN_{11}}$	1.2	**0.9**	1.2	**0.9**	**0.9**
d_{35}	XN_{12}	0.75	0.75	0.75	0.75	**0.6**
d_{36}	XN_{13}	0.75	0.75	0.75	0.75	**0.6**
d_{37}	XN_{14}	0.75	0.75	0.75	0.75	**0.6**
d_{38}	XN_{15}	1.0	1.0	1.0	1.0	**0.8**
d_{39}	XN_{16}	1.14	1.14	1.14	1.14	**0.9083**
d_{40}	XN_{17}	1.2	1.2	1.2	1.2	**0.9**
d_{41}	XN_{18}	1.4	1.4	1.4	1.4	**1.1**
d_{42}	XN_{19}	1.2	1.2	1.2	1.2	**0.9**
d_{43}	XN_{20}	1.4	1.4	1.4	1.4	**1.1**
d_{44}	XN_{21}	2.13	2.13	2.13	2.13	**1.7**

Table 19. Cost, Probabilities of Failure at RBDO Optimum Design, and Optimization Details

Performance Measure	Based on Experience	Variable Screening	Variable Screening + Cost Function	Full Dimension
Cost	249.10	259.83	244.17	225.68
G_1	10.06%	9.94%	9.96%	9.96%
G_2	10.03%	10.11%	10.04%	9.99%
G_3	0.00%	0.00%	0.00%	0.00%
G_4	0.11%	0.11%	0.09%	0.10%
G_5	1.96%	1.95%	2.04%	1.91%
G_6	9.95%	10.01%	10.02%	9.99%
G_7	9.89%	9.93%	9.95%	10.01%
G_8	10.03%	9.98%	9.91%	9.07%
G_9	10.04%	9.99%	9.87%	9.92%
G_{10}	0.00%	0.00%	0.00%	0.00%
G_{11}	10.02%	9.97%	9.93%	10.02%
No. of Design Iter.	20	30	20	21
No. of CAEs	2,666	4,358	3,306	-
No. of Black Box Calls	473	727	573	-

Table 20. RBDO Optimum Design with I-RBDO and True Model

Design Variables	Corresponding RVs	Based on Experience		Variable Screening		Var. Screen. + Cost Fn.	
		I-RBDO	44-D Surrogate	I-RBDO	44-D Surrogate	I-RBDO	44-D Surrogate
d_1	X_1	1.8343	1.8371	1.8366	1.8359	1.8336	1.8425
d_2	X_2	2.1810	2.1799	2.1804	2.1807	2.1806	2.1771
d_3	X_3	2.8528	2.8578	2.8561	2.8576	2.8540	2.8654
d_4	X_4	1.9817	1.9807	1.9810	1.9851	1.9856	1.9525
d_5	X_5	2.7195	2.7222	2.7228	2.7233	2.7261	2.7209
d_6	X_6	2.2543	2.2488	2.2497	2.2501	2.2558	2.2464
d_7	X_7	2.3199	2.3173	2.3185	2.3169	2.3207	2.3265
d_8	X_8	1.7904	1.8	1.7966	1.7985	1.786	1.8
d_9	X_{10}	-	-	0.9	0.9	0.9	0.9
d_{18}	X_{20}	0.9	0.9	0.9	0.9	0.9	0.9
d_{19}	X_{23}	-	-	0.6	0.6	0.6	0.6
d_{21}	X_{25}	0.6897	0.6875	0.5429	0.5424	0.5871	0.5826
d_{22}	X_{26}	-	-	1.1	1.1	1.1	1.1
d_{24}	XN_1	0.7	0.7	0.7	0.7	0.7	0.7
d_{27}	XN_4	-	-	-	-	0.7	0.7
d_{32}	XN_9	0.6	0.6	-	-	0.6	0.6
d_{33}	XN_{10}	0.6	0.6	-	-	0.6	0.6
d_{34}	XN_{11}	0.9	0.9	-	-	0.9	0.9

Table 21. Reliability Analysis Result Using Full-Dimensional Surrogate Model

Performance Measure	Baseline Design	Based on Experience	Variable Screening	Variable Screening + Cost Function	Full Dimension
Cost	269.47	249.10	259.83	244.17	225.68
G_1	48.25%	10.06%	9.96%	10.00%	10.05%
G_2	51.34%	10.02%	10.11%	10.04%	10.09%
G_3	54.14%	0.00%	0.00%	0.00%	0.00%
G_4	55.57%	0.10%	0.12%	0.09%	0.12%
G_5	58.94%	1.96%	1.93%	1.98%	1.91%
G_6	59.70%	10.08%	10.05%	10.05%	10.00%
G_7	59.86%	10.20%	10.04%	9.91%	10.06%
G_8	53.23%	10.02%	10.03%	9.97%	9.14%
G_9	51.15%	10.02%	9.96%	9.96%	9.96%
G_{10}	49.10%	0.00%	0.00%	0.00%	0.00%
G_{11}	52.46%	17.70%	11.23%	11.17%	10.05%

5. Conclusion

A new efficient and effective variable screening method for RBDO is proposed in this paper. For the proposed methods, the output variance is used as a measure that can identify important design variables. Thus, a partial output variance based on the univariate DRM is proposed to approximate the output variance efficiently and to identify the design variables that affect output variance more significantly than others. The univariate DRM and partial output variance only require multiple 1-D surrogate models, which is much more efficient than the full-dimensional surrogate models. Hence, the proposed method has great merit in efficiency as well as effectiveness. To reduce computational time and maintain a user-specified statistical error level, hypothesis testing is used in the variable screening process. Also, a required minimum number of samples for calculating the correct output variance is proposed using the user-specified error level. In addition, the quadratic interpolation method is tailored to be applied to efficient partial output variance calculation.

Two analytical examples and a 44-D industrial example are used to verify the performance of the proposed variable screening method. Through the analytical examples, it is shown that at least quadratic approximation is required for the 1-D surrogate model and that partial output variance is a good measure that successfully identifies important variables. In the industrial example, 14 design variables out of 44 are selected by considering the output variances of 11 constraints. For comparison, another 14 design variables selected based on experience are used. In addition, 18 design variables are selected by adding four design variables, which affect the objective function significantly while not affecting the output variances much, to the 14 design variables previously selected with the proposed method. The selection based on experience shows a 7.6% reduced cost value, whereas the target probability of failure is violated by 77%. However, selection by the proposed method shows only a 12.3% disagreement of target value and a 3.6% reduced cost value. Moreover, the selection of 18 design variables shows 11.7% target disagreement as well as 9.4% reduced cost value. Therefore, the performance of the proposed variable screening method is verified.

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